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Integrability and exact spectrum of a pairing model for nucleons

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Abstract
A pairing model for nucleons, introduced by Richardson in 1966, which describes proton–neutron pairing as well as proton–proton and neutron–neutron pairing, is re-examined in the context of the quantum inverse scattering method. Specifically, this shows that the model is integrable by enabling the explicit construction of the conserved operators. We determine the eigenvalues of these operators in terms of the Bethe ansatz, which in turn leads to an expression for the energy eigenvalues of the Hamiltonian.

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1. Introduction

Pairing model Hamiltonians have again become the focus of many theoretical condensed matter investigations due to the fact that the experimental work on metallic nanoparticles (also referred to as small metallic grains) has detected evidence of pairing interactions [1]. In order to gain an insight into the physical properties of small metallic grains, substantial attention has been devoted to the analysis of the (reduced) BCS Hamiltonian which is believed to be appropriate to describe the dynamics of these systems [2]. An important point in this regard is that the treatment originally proposed by Bardeen, Cooper and Schrieffer for bulk systems, using variational wavefunctions with an undetermined number of particles (grand canonical ensemble), is not applicable to the study of a small metallic grain where the number of electrons remains fixed (canonical ensemble). This aspect has generated activity in analysing the BCS Hamiltonian under this constraint [3]. Remarkably, an exact solution of the reduced BCS Hamiltonian was obtained some time ago in a series of works by Richardson and Sherman [4]. In these papers, the BCS Hamiltonian was studied for the purpose of application to pairing interactions in nuclear systems, and as such the solution escaped the attention of the condensed matter physics community for a considerable time. More recently, it was shown by Cambiaggio, Rivas and Saraceno [5] that the model is also integrable, i.e. there exists a

set of mutually commuting operators which commute with the Hamiltonian. These features can be reproduced in the framework of the quantum inverse scattering method (QISM) using a solution of the Yang–Baxter equation as shown in [6, 7]. This approach has the significant advantage that the computation of form factors and correlation functions can be undertaken in this algebraic framework [6, 8].

In the papers [9, 10] Richardson introduced a coupled pairing model for nuclear systems which accommodates proton–neutron pairing interactions as well as the proton–proton and neutron–neutron couplings. This model was recently studied in [11]. In this paper, we will show that the techniques employed in [6, 7] can be applied to this model to show that it is integrable, and for the determination of the energy spectrum. This formulation also opens the possibility for the calculation of form factors and correlation functions by algebraic means, in analogy with the results of [6]. For the present case, the solution is obtained through the use of a solution of the Yang–Baxter equation associated with the Lie algebra $so(5)$. The Hamiltonian has the explicit form

$$H = \sum_{j}^{D} \epsilon_j n_j - g \sum_{j,k}^{D} (b_j^\dagger(1) b_k(1) + b_j^\dagger(2) b_k(2) + b_j^\dagger(3) b_k(3)) \tag{1}$$

where g is an arbitrary coupling parameter and D is the total number of distinct energy levels. Above, n_j is the number operator for paired nucleons at energy level ϵ_j and $b(i)_j, b_j^\dagger(i), i = 1, 2, 3$ are the annihilation and creation operators for three sets of generalized, non-commuting, hard-core boson operators satisfying the relations (amongst others)

$$(b_j^\dagger(i))^2 = 0 \qquad i = 1, 2 \qquad (b_j^\dagger(3))^3 = 0$$
$$[b_j(i), b_k(l)] = [b_j^\dagger(i), b_k^\dagger(l)] = [b_j(i), b_k^\dagger(l)] = 0 \qquad \text{for} \quad k \neq j.$$

The three sets of hard-core boson operators correspond to proton–proton, neutron–neutron and proton–neutron pairing in a nuclear system. Their explicit forms will now be made clear. Two sets of two-fold degenerate Fermi operators $c_\pm$ and $d_\pm$ and their Hermitian conjugates are introduced. $c_\pm$ and $d_\pm$ represent the protons and neutrons, the subscripts $\pm$ referring to time-reversed states. The hard-core bosons are realized by

$$b^\dagger(1) = c_+^\dagger c_-^\dagger \qquad b^\dagger(2) = d_+^\dagger d_-^\dagger \qquad b^\dagger(3) = \frac{1}{\sqrt{2}}(c_+^\dagger d_-^\dagger + d_+^\dagger c_-^\dagger)$$

with appropriate definitions for the Hermitian conjugates. The Hamiltonian (1) corresponds to a special case where the energy level spacings for the protons and neutrons are the same and the scattering coupling is the same for proton–proton, neutron–neutron and proton–neutron pairings. In this instance, the Hamiltonian acquires an *isospin* symmetry which plays an important role in our analysis below.

The energy levels ϵ_j are degenerate. Each level can be empty or occupied by protons and/or neutrons in two-fold degenerate time-reversed states. This gives the degeneracy of each level as $2^4 = 16$. However, the above Hamiltonian only scatters paired nucleons giving rise to a blocking effect (cf, the blocking effect for the BCS model discussed in [2]). For any energy level which contains an odd number of nucleons, the pairing interaction acts trivially, and these states can be discarded from the Hilbert space. (Hence, we choose n_j to count the number of *paired* nucleons at ϵ_j, rather than the number of nucleons.) Furthermore, it is assumed that the proton–neutron pairing is between time-reversed states which are symmetric under interchange of protons and neutrons. In this case, the pairing interaction is also trivial on the nontime-reversed paired proton–neutron states and the antisymmetric time-reversed paired proton–neutron state, giving rise to a five-dimensional space at each level ϵ_j on which the

scattering is non-trivial. (In the language of [9], this Hilbert space is spanned by the *seniority-zero* states. In the subsequent paper [10], Richardson extended his results to seniority-one and -two states. However, we will not consider these cases here.) It is convenient to use the fundamental representation of the $so(5)$ Lie algebra to construct the local operators acting on each of these spaces, which we will now discuss.

2. The Lie algebra *so*(5)

We can construct the fundamental representation of the $so(5)$ Lie algebra in the following manner. Define $\overline{m} = 6 - m$. For 5×5 matrices, consider the subset

$$a_n^m = e_{\overline{n}}^m - e_{\overline{m}}^n = -a_m^n$$

where e_n^m denotes the matrix with 1 in the (m, n) position and zeros elsewhere. Note that $a_m^m = 0$. We will denote the five-dimensional space on which these operators act by V. The operators a_n^m close to form the fundamental or defining representation of the Lie algebra $so(5)$ with commutation relations

$$\left[a_n^m, a_l^p\right] = \delta_{\overline{n}}^p a_l^m + \delta_{\overline{m}}^l a_p^n + \delta_l^n a_m^p + \delta_{\overline{p}}^m a_n^l \tag{2}$$

A basis for the Lie algebra is given by the set

$$\left\{a_n^m : 1 \leqslant m < n \leqslant 5\right\}$$

which gives ten linearly independent generators. Explicitly, the basis generators read

$$\begin{array}{ll} a_2^1 = e_4^1 - e_5^2 & a_3^1 = e_3^1 - e_5^3 \\ a_4^1 = e_2^1 - e_5^4 & a_5^1 = e_1^1 - e_5^5 \\ a_3^2 = e_3^2 - e_4^3 & a_4^2 = e_2^2 - e_4^4 \\ a_5^2 = e_1^2 - e_4^5 & a_4^3 = e_2^3 - e_3^4 \\ a_5^3 = e_1^3 - e_3^5 & a_5^4 = e_1^4 - e_2^5. \end{array}$$

Note that the representation is unitary, and specifically

$$\left(a_n^m\right)^\dagger = a_{\overline{m}}^{\overline{n}}. \tag{3}$$

Next, we recall some established results on the representation theory of $so(5)$. For a more detailed discussion, see for example [12]. The finite-dimensional irreducible representations of $so(5)$ are uniquely determined by the highest weight labels Λ_1, Λ_2 which are the eigenvalues of the Cartan elements

$$h_1 = a_5^1 \qquad h_2 = a_4^2$$

acting on the highest weight state. (These operators are self-adjoint and mutually commuting and so can be diagonalized simultaneously.) The highest weight state is the unique vector v which vanishes under the action of the raising operators, viz,

$$a_2^1 v = a_3^1 v = a_4^1 v = a_3^2 v = 0.$$

The highest weight labels Λ_1, Λ_2 take integer or half odd-integer values and are subject to the conditions

$$\Lambda_1 \geqslant \Lambda_2 \geqslant 0 \qquad \Lambda_1 - \Lambda_2 \in \mathbb{Z}.$$

In the case of the fundamental representation, we have $\Lambda_1 = 1$, $\Lambda_2 = 0$ and the highest weight vector is $|1\rangle \equiv (1, 0, 0, 0, 0)^t$, with t the matrix transposition operation.

The $so(5)$ algebra admits a second-order Casimir invariant

$$C = \sum_{m,n}^{5} a_n^m a_{\overline{m}}^{\overline{n}} \tag{4}$$

commuting with all elements of $so(5)$, which can be verified explicitly from the commutation relations (2). Due to Schur's lemma, on each irreducible finite-dimensional representation the Casimir element (4) takes a constant eigenvalue, which is given by

$$\chi_C(\Lambda_1, \Lambda_2) = 2(\Lambda_1(\Lambda_1 + 3) + \Lambda_2(\Lambda_2 + 1)).$$

For the fundamental representation we clearly have $\chi_C(1, 0) = 8$.

An important ingredient in the following construction is the existence of a canonical $so(3)$ subalgebra spanned by the basis elements

$$L^0 = a_4^2 \qquad L^+ = a_3^2 \qquad L^- = a_4^3.$$

The Casimir operator for this $so(3)$ subalgebra is given by

$$K = L^+L^- + L^-L^+ + (L^0)^2.$$

The irreducible finite-dimensional representations of the $so(3)$ algebra have a unique highest weight vector w which satisfies $L^+w = 0$. These representations are uniquely characterized by the eigenvalue μ of L^0 acting on w. The allowable values of μ are such that it is a non-negative integer or half odd-integer. The eigenvalue of K on such a representation is given by

$$\chi_K(\mu) = \mu(\mu + 1).$$

The realization of this canonical $so(3)$ subalgebra that we give below is referred to as the isospin algebra in [9].

3. The Yang–Baxter equation and integrability

The basis for constructing an integrable model through the QISM [13] is a solution of the Yang–Baxter equation,

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v) \tag{5}$$

which is a matrix solution acting on a three-fold tensor product space $V \otimes V \otimes V$. Above, the subscripts refer to which two of the three spaces the operator $R(u) \in \mathrm{End}(V \otimes V)$ acts upon. Solutions of the Yang–Baxter equation associated with representations of Lie algebras are well known. The R-matrix solution associated with the fundamental representation of $so(5)$ discussed above takes the following form [14]: set

$$I = \sum_{m,n}^{5} e_m^m \otimes e_n^n \qquad P = \sum_{m,n}^{5} e_n^m \otimes e_m^n \qquad Q = \sum_{m,n}^{5} e_n^m \otimes e_{\overline{n}}^{\overline{m}}.$$

Then

$$R(u) = I + \frac{2\eta}{u}P - \frac{2\eta}{u + 3\eta}Q$$

provides a solution of (5) with η being a free parameter. This solution is $so(5)$ invariant in that

$$[R(u), I \otimes x + x \otimes I] = 0$$

for any $x \in so(5)$. We note the following properties:
(1) Unitarity

$$R(u)R(-u) = \left(\frac{u^2 - 4\eta}{u^2}\right) I \otimes I$$

(2) Crossing symmetry

$$R^{t_1}(-u - 3\eta) = (I \otimes A)R(u)(I \otimes A)$$

where A is the matrix with elements $A_n^m = \delta_{\bar{n}}^m$ and t_1 denotes matrix transposition in the first space of the tensor product.

By the usual procedure of the QISM, we define a transfer matrix acting on the D-fold tensor product space via

$$t(u) = \mathrm{Tr}_0\,(G_0 R_{0D}(u - \epsilon_D) \cdots R_{01}(u - \epsilon_1))$$

which gives a commuting family satisfying $[t(u), t(v)] = 0$. Above, Tr_0 denotes the trace taken over an auxiliary space labelled by 0, and G can be any matrix which satisfies

$$[R(u), G \otimes G] = 0.$$

We choose $G = \exp\left(\alpha\eta a_5^1\right)$ for this particular model. Using either the analytic Bethe ansatz, which exploits the unitarity and crossing symmetry properties [15], or the algebraic Bethe ansatz developed by Martins and Ramos for the $so(n)$ series [16], the eigenvalues of the transfer matrix are found to be

$$\Lambda(u) = \exp(\alpha\eta) \prod_k^D \frac{(u - \epsilon_k + 2\eta)}{(u - \epsilon_k)} \prod_i^M \frac{(u - v_i - \eta)}{(u - v_i + \eta)} + \exp(-\alpha\eta) \prod_k^D \frac{(u - \epsilon_k + \eta)}{(u - \epsilon_k + 3\eta)} \prod_i^M \frac{(u - v_i + 4\eta)}{(u - v_i + 2\eta)} + \Lambda_0(u)$$

where $\Lambda_0(u)$ are the transfer matrix eigenvalues for the R-matrix associated with the spin-1 Babujian–Tahktajan model [17], with inhomogeneities v_i. These eigenvalues read

$$\Lambda_0(u) = \prod_i^M \frac{(u - v_i + 3\eta)}{(u - v_i + \eta)} \prod_j^N \frac{(u - w_j)}{(u - w_j + 2\eta)} + \prod_i^M \frac{(u - v_i)}{(u - v_i + 2\eta)} \prod_j^N \frac{(u - w_j + 3\eta)}{(u - w_j + \eta)} + \prod_j^N \frac{(u - w_j + 3\eta)}{(u - w_j + \eta)} \frac{(u - w_j)}{(u - w_j + 2\eta)}.$$

The parameters v_i, w_j are required to satisfy the Bethe ansatz equations,

$$\exp(\alpha\eta) \prod_k^D \frac{(v_j - \epsilon_k + \eta)}{(v_j - \epsilon_k - \eta)} = -\prod_l^N \frac{(v_j - w_l - \eta)}{(v_j - w_l + \eta)} \prod_i^M \frac{(v_j - v_i + 2\eta)}{(v_j - v_i - 2\eta)} \tag{6}$$

$$\prod_i^M \frac{(w_j - v_i - \eta)}{(w_j - v_i + \eta)} = -\prod_k^N \frac{(w_j - w_k - \eta)}{(w_j - w_k + \eta)}. \tag{7}$$

Define the operators

$$T_j = \lim_{u \to \epsilon_j} \frac{u - \epsilon_j}{2\eta} t(u) \tag{8}$$

which satisfy

$$[T_j, T_k] = 0. \tag{9}$$

Explicitly

$T_j = G_i R_{jD}(\epsilon_j - \epsilon_D) \cdots R_{j(j+1)}(\epsilon_j - \epsilon_{j+1}) R_{j(j-1)}(\epsilon_j - \epsilon_{j-1}) \cdots R_{j1}(\epsilon_j - \epsilon_1).$

Now, by taking the *quasi-classical* expansion,

$$T_j = I + \eta\tau_j + o(\eta^2) \tag{10}$$

we find

$$\tau_j = \alpha\psi_j + 2\sum_{k\neq j}^{D} \frac{\phi_{jk}}{\epsilon_j - \epsilon_k}$$

where

$$\phi = \sum_{m,n}^{5} e_n^m \otimes a_{\overline{m}}^n$$

and for ease of notation we set $\psi = a_5^1$. Setting

$$\begin{aligned}
\theta &= \sum_{m,n}^{5} a_n^m \otimes a_{\overline{m}}^{\overline{n}} \\
&= \sum_{m,n}^{5} \left(e_{\overline{n}}^m - e_{\overline{m}}^n\right) \otimes a_{\overline{m}}^{\overline{n}} \\
&= \sum_{m,n}^{5} e_n^m \otimes a_{\overline{m}}^n + \sum_{m,n} e_{\overline{m}}^n \otimes a_{\overline{n}}^{\overline{m}} \\
&= 2\sum_{m,n}^{5} e_n^m \otimes a_{\overline{m}}^n \\
&= 2\phi
\end{aligned}$$

shows that we may write

$$\tau_j = \alpha\psi_j + \sum_{k\neq j}^{D} \frac{\theta_{jk}}{\epsilon_j - \epsilon_k} \tag{11}$$

which satisfy $[\tau_j, \tau_k] = 0$ in view of (9) and (10). The conserved operators τ_j provide a set of generalized Gaudin Hamiltonians [18]. The Gaudin Hamiltonians are recovered in the limit $\alpha \to 0$.

Consider the action of the $so(3)$ Casimir on the D-fold tensor product space,

$$K \to \sum_{i,j}^{D} \left(L_i^+ L_j^- + L_i^- L_j^+ + L_i^0 L_j^0\right).$$

It is easily deduced that

$$[K, \psi_j] = 0.$$

When $\alpha = 0$ (and so $G = I$),

$$[K, \tau_j] = 0$$

since in this instance the operators τ_j are $so(5)$ invariant as a consequence of the $so(5)$ invariance of the R-matrix. We thus see that

$$[K, \tau_j] = 0$$

in general. The set of operators $\{\tau_i, K\}$ are mutually commutative and so can be used to define an integrable Hamiltonian through any function of these operators. With an appropriate choice, we show below that the pairing Hamiltonian (1) introduced in [9] can be reproduced, thus establishing the integrability of this model.

4. Pairing Hamiltonian

First, let us realize the five-dimensional space V in terms of the two sets of two-fold degenerate Fermi operators $c_\pm$, $c_\pm^\dagger$ and $d_\pm$, $d_\pm^\dagger$ introduced earlier. We make the identifications

$$|1\rangle = |0\rangle \qquad |2\rangle = d_+^\dagger d_-^\dagger|0\rangle \qquad |3\rangle = \frac{1}{\sqrt{2}}(c_+^\dagger d_-^\dagger + d_+^\dagger c_-^\dagger)|0\rangle$$
$$|4\rangle = c_-^\dagger c_+^\dagger|0\rangle \qquad |5\rangle = c_+^\dagger c_-^\dagger d_+^\dagger d_-^\dagger|0\rangle.$$

Set $n^c = c_+^\dagger c_+ + c_-^\dagger c_-$, $n^d = d_+^\dagger d_+ + d_-^\dagger d_-$ and $n = 1/2(n^c + n^d)$, which measures the number of paired fermions. We have the following realization of the $so(5)$ generators,

$$a_5^4 = c_-^\dagger c_+^\dagger \qquad a_5^2 = d_+^\dagger d_-^\dagger \qquad a_4^3 = \frac{1}{\sqrt{2}}(c_-^\dagger d_- + c_+^\dagger d_+)$$
$$a_5^3 = \frac{1}{\sqrt{2}}(c_+^\dagger d_-^\dagger + d_+^\dagger c_-^\dagger) \qquad a_5^1 = I - \tfrac{1}{2}(n^c + n^d) \qquad a_4^2 = \tfrac{1}{2}(n^d - n^c)$$

and

$$a_4^1 = \left(a_5^2\right)^\dagger \qquad a_3^1 = \left(a_5^3\right)^\dagger \qquad a_3^2 = \left(a_4^3\right)^\dagger \qquad a_2^1 = \left(a_5^4\right)^\dagger.$$

The representation of the canonical $so(3)$ subalgebra generated by $\{L^+, L^-, L^0\}$ is the isospin algebra of [9], and the operator ψ is a $U(1)$ generator. We can also identify the generalized hard-core boson operators with certain elements of the $so(5)$ algebra through

$$b^\dagger(1) = -a_5^4 \qquad b^\dagger(2) = a_5^2 \qquad b^\dagger(3) = a_5^3$$

and corresponding relations for the Hermitian conjugates from (3). We may now express

$$\tfrac{1}{2}\theta = b^\dagger(1)\otimes b(1) + b^\dagger(2)\otimes b(2) + b^\dagger(3)\otimes b(3) + b(1)\otimes b^\dagger(1) + b(2)\otimes b^\dagger(2)$$
$$+ b(3)\otimes b^\dagger(3) + L^+\otimes L^- + L^-\otimes L^+ + L^0\otimes L^0 + \psi\otimes\psi.$$

Define the Hamiltonian

$$\begin{aligned} H &= \frac{-1}{\alpha}\left(\sum_j^D\left(\epsilon_j - \frac{3}{\alpha}\right)\tau_j - K + 4D\right) + \frac{1}{\alpha^3}\sum_{j,k}^D \tau_j\tau_k + \sum_j^D \epsilon_j \\ &= -\sum_j^D\left(\epsilon_j - \frac{3}{\alpha}\right)\psi_j - \frac{1}{2\alpha}\sum_j^D\sum_{k\neq j}^D \theta_{jk} \\ &\quad + \frac{1}{\alpha}\left(\sum_{j,k}^D \psi_j\psi_k + K - 4D\right) + \sum_j^D \epsilon_j \\ &= -\sum_j^D\left(\epsilon_j - \frac{3}{\alpha}\right)\psi_j - \frac{1}{2\alpha}\sum_{j,k}^D \theta_{jk} + \frac{1}{\alpha}\left(\sum_{j,k}^D \psi_j\psi_k + K\right) + \sum_j^D \epsilon_j \end{aligned}$$

where in the last line we have used the fact that the $so(5)$ Casimir invariant C takes the eigenvalue 8 in the five-dimensional representation, as mentioned earlier. Expressing the $so(5)$ elements in terms of the hard-core boson operators as indicated above yields (1) with $g = 2/\alpha$. The Hamiltonian describes two coupled identical BCS models, where, in addition to the customary pairing (characterized by the operators $b^\dagger(1), b(1), b^\dagger(2), b(2)$), fermions from each BCS system at the same energy level ϵ_j and in time-reversed states can pair in such a fashion that the wavefunction is symmetric under interchange of the two BCS systems (described by $b^\dagger(3), b(3)$). This shows that the Hamiltonian has a natural interpretation as a pairing model for nucleons which includes proton–neutron pairing.

5. Energy spectrum

It remains to determine the eigenvalues of the Hamiltonian, which is achieved by computing the eigenvalues of the conserved operators. The eigenstates can be labelled by the eigenvalues of the Cartan elements $h_1 = a_5^1 = \Psi$, $h_2 = a_4^2 = L^0$ acting on the tensor product space. For each value of M and N which appear in the solution of the Bethe ansatz equations, we find that the corresponding eigenvalues of the Cartan elements are

$$\Lambda_1 = D - M \qquad \Lambda_2 = M - N = \mu$$

or equivalently

$$n = M \qquad n^d - n^c = 2(M - N).$$

We begin with the operator K. It follows from the algebraic construction of the Bethe states due to Martins and Ramos [16] that each Bethe state is a highest weight state with respect to the $so(3)$ subalgebra. (When $\alpha = 0$ the Bethe states are highest weight states with respect to the full $so(5)$ algebra, but generic values of α break this symmetry.) Since K is simply the $so(3)$ Casimir operator, it takes the eigenvalue $\chi_K(M - N) = (M - N)(M - N + 1)$ on such a Bethe state.

From (8) and (10) we see that the eigenvalues for τ_j can be obtained from the quasi-classical limit of the transfer matrix eigenvalues. They read

$$\lambda_j = \alpha + \sum_{k\neq j}^{D} \frac{2}{\epsilon_j - \epsilon_k} - \sum_{i}^{M} \frac{2}{\epsilon_j - v_i} \tag{12}$$

and from the limit of the Bethe ansatz equations (6) and (7) we obtain

$$\alpha + \sum_{k}^{D} \frac{2}{v_j - \epsilon_k} = \sum_{i\neq j}^{M} \frac{4}{v_j - v_i} + \sum_{l}^{N} \frac{2}{w_l - v_j} \qquad \sum_{i}^{M} \frac{1}{w_j - v_i} = \sum_{k\neq j}^{N} \frac{1}{w_j - w_k}. \tag{13}$$

Using these Bethe ansatz equations, we can derive the following identities:

$$\sum_{j}^{N}\sum_{i}^{M} \frac{1}{w_j - v_i} = \sum_{j}^{N}\sum_{k\neq j}^{N} \frac{1}{w_j - w_k} = 0$$

$$\alpha M + \sum_{j}^{M}\sum_{k}^{D} \frac{2}{v_j - \epsilon_k} = \sum_{j}^{M}\sum_{i\neq j}^{M} \frac{4}{v_i - v_j} + \sum_{j}^{M}\sum_{l}^{N} \frac{2}{w_l - v_j} = 0$$

$$\sum_{j}^{M}\sum_{l}^{N} \frac{w_l}{w_l - v_j} - \sum_{j}^{M}\sum_{l}^{N} \frac{v_j}{w_l - v_j} = MN$$

$$\sum_{j}^{N}\sum_{i}^{M} \frac{w_j}{w_j - v_i} = \sum_{j}^{N}\sum_{k\neq j}^{N} \frac{w_j}{w_j - w_k} = \tfrac{1}{2}N(N-1)$$

$$\sum_{j}^{M}\sum_{k}^{D} \frac{v_j}{v_j - \epsilon_k} - \sum_{j}^{M}\sum_{k}^{D} \frac{\epsilon_k}{v_j - \epsilon_k} = ML$$

$$\alpha \sum_{j}^{M} v_j + \sum_{j}^{M}\sum_{k}^{D} \frac{2v_j}{v_j - \epsilon_k} = \sum_{j}^{M}\sum_{i\neq j}^{M} \frac{4v_j}{v_j - v_i} + \sum_{j}^{M}\sum_{l}^{N} \frac{2v_j}{w_l - v_j}$$
$$= 2M(M-1) - 2MN + N(N-1).$$

We now obtain from these identities

$$\sum_j^D \lambda_j = \alpha(D - M)$$

$$\sum_j^D \epsilon_j \lambda_j = \alpha \sum_j^D \epsilon_j + D(D-1) - \alpha \sum_j^M v_j + 2M(M-1) + N(N-1) - 2M(D+N).$$

Using these results, we find that the energies are given by

$$E = \frac{-1}{\alpha}\left(\sum_j^D \epsilon_j \lambda_j - 3(D-M) - (M-N)(M-N+1) + 4D\right) + \frac{1}{\alpha^3}\sum_{j,k}^D \lambda_j \lambda_k + \sum_j^D \epsilon_j = \sum_j^M v_j.$$

This energy expression shows that the roots $\{v_j\}$ of the Bethe ansatz equations are simply the quasi-particle excitation energies. It is interesting to note in the case $N = 0 \Rightarrow n^c = 0$, where the model describes a single reduced BCS system since there is only one type of nucleon, the Bethe ansatz equations and energy expression coincide with those obtained by Richardson and Sherman [4].

Finally, it is necessary to compare our results with those obtained by Richardson [9], in which the following Bethe ansatz equations were obtained,

$$\frac{1}{g} + \sum_k^D \frac{1}{v_j - \epsilon_k} = \frac{M(M-3) + (M-N)(M-N+1)}{M(M-1)} \sum_{i \neq j}^M \frac{1}{v_j - v_i} \tag{14}$$

and are obviously different from our results. The explanation for this difference stems from the fact that the ansätze adopted for the eigenstate wavefunctions are different in each case. Richardson chose wavefunctions which have eigenvalue zero under the action of the isospin operator L^0. As we have indicated above, the states we construct are highest weight states with respect to the isospin algebra. An important open problem is to prove the equivalence of these two solutions.

6. Conclusion

We have shown, by using the QISM, that the coupled BCS Hamiltonian proposed by Richardson [9] to accommodate proton–neutron pairing in nuclear systems is integrable. We have also determined expressions for the energy eigenvalues of the model in terms of a Bethe ansatz solution of coupled equations. It should be emphasized that although the model studied here is based on a specific Lie algebra and representation, the construction that we have employed to demonstrate integrability is entirely general. It can be equally applied to any representation of any Lie algebra or superalgebra, to yield a vast class of integrable systems. For a recent example based on the spin-1 representation of the $so(3)$ algebra see [19].

An interesting question to ask is whether this solution is complete, i.e. are all energy levels obtained? It is well known that for many Bethe ansatz solvable models where there is an underlying Lie algebra symmetry the eigenstates are highest weight states with respect to this algebra [20–22]. By computing the dimensions of each multiplet generated by these highest weight states and then employing a combinatorial argument, completeness can be proved.

For the present model, where the R-matrix solution has $so(5)$ symmetry, this symmetry is broken in the construction of the transfer matrix by the inclusion of the operator G, and

an $so(3)$ symmetry (isospin) remains for the conserved operators τ_j. The degeneracies of the eigenvalues can be classified in terms of $so(3)$ multiplets. In the $\alpha \to 0$ limit, the $so(5)$ symmetry is restored and we find that the Bethe states are $so(5)$ highest weight states, so there is an increase in the degeneracy of each eigenvalue at this point, or equivalently, a decrease in the number of distinct eigenvalues. Fortunately, the Bethe ansatz equations (13) admit more solutions for generic values of α than the $\alpha = 0$ case and automatically accommodate this facet. This is easily illustrated in the instance $D = 2$, $M = 1$, $N = 0$, in which case we need only solve

$$\alpha + \frac{2}{v - \epsilon_1} + \frac{2}{v - \epsilon_2} = 0. \tag{15}$$

For non-zero α, this is a quadratic equation with two solutions for v. When α is zero, the equation is linear with the unique finite solution $v = (\epsilon_1 + \epsilon_2)/2$. (The equation is also satisfied by $v = \infty$ which is the limit of one solution of (15) as $\alpha \to 0$. However, such infinite solutions are trivial in the sense that they do not contribute to the eigenvalues (12) for the conserved operators.) For general values of D with $M = 1, N = 0$, (13) gives rise to a polynomial equation of order D for $\alpha \neq 0$, but this equation reduces to order $(D - 1)$ when $\alpha = 0$. Whether the Bethe ansatz solutions give the complete spectrum for generic α is an open problem still to be solved, but the discussion above shows that it is possible since the breaking of the $so(5)$ symmetry to $so(3)$ is accompanied by an increase in the number of solutions to the Bethe ansatz equations. For the purpose of counting the states, the results discussed in [23] may be appropriate.

A final aspect for consideration is the possibility to compute form factors and correlation functions for this model. By rederiving the solution in the framework of the QISM, we hope to motivate further studies that are necessary for this task, such as an analogue of Slavnov's formula for wavefunction scalar products, which is well known for $su(2)$ models [24].

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